

A MATHEMATICAL DEFINITION OF

COULOMB BRANCHES OF

3d  $N=4$  SUPERSYMMETRIC GAUGE THEORIES

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## § 1. Introduction

### Theoretical Physics

$G_c$  : compact Lie group (say  $G_c = U(n)$ )

$M$  : quaternionic representation (say  $M = \mathbb{H}^n = \mathbb{C}^n \otimes (\mathbb{C}^n)^*$ )

→ 3d N=4 SUSY gauge theory

lagrangian defined for  $\begin{cases} G_c\text{-principal bundle } P \text{ on a 3d Riemannian manifold} \\ \text{connection } A \quad P \times^{G_c} M \text{-valued spinor } s \\ + \text{ SUSY partners} \quad \text{(fields)} \end{cases}$

quantization → physical quantities (correlation functions etc)

Hilbert spaces etc

(This procedure (e.g. Feynman path integral  $\int \mathcal{D}A \mathcal{D}s \dots \star$ )  
is not mathematically rigorous. ALL FIELDS

## Seiberg-Witten (1996)

Constant branch (a branch of vacua = space of fields where lagrangian takes minimum)  
 receives "quantum corrections"

$$\mathcal{M}_C^{\text{classical}} = (\mathbb{R}^3 \times S^1)^{\text{rank } G_C} / \text{Weyl group}$$

(S=0, A: trivial)

$\rightsquigarrow \mathcal{M}_C$ : "correct" space reflecting QFT features of the gauge theory

$\mathcal{M}_C$  is (claimed to be) a hyperKähler manifold, possibly with singularities.

Physicists (e.g. Hanany-Witten) subsequently determined  $\mathcal{M}_C$  in various examples of  $G_C, M$ . (via **mathematically unjustified** techniques)

Remark There is a similar story for 4d N=2 SUSY gauge theory.  
 A Gauge theory is controlled by an integrable system.

Seiberg-Witten curves = spectral curves

$G = \text{complexification of } G_c$  e.g.  $G_c = \text{U}(n) \Rightarrow G = \text{GL}(n, \mathbb{C})$

Problem. Define  $M_c \equiv M_c(G, M)$  from  $(G, M)$  in a mathematically rigorous way.

When  $M = N \oplus N^*$  (cotangent type)

$M_c$  is regarded as a holomorphic symplectic variety.

An Answer was given by Braverman - Finkelberg - Nakajima 2018

more general  $M$  : Braverman - Dhillon - Finkelberg - Raskin - Trushin  
Teleman

This construction is based on techniques of geometric representation theory,  
in particular, convolution algebras.

It recovers numerous examples, which were calculated by physicists.

It also gives new examples.

## §2. Construction

We construct  $\mathbb{C}[[M_C]] = \text{ring of (algebraic) holomorphic functions on } M_C$ .

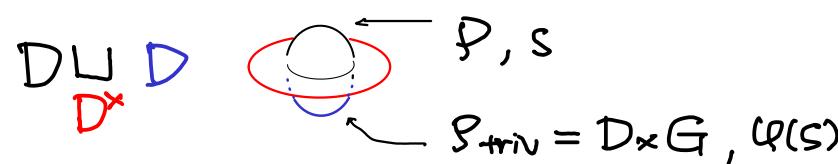
Then  $M_C = \text{Spec } \mathbb{C}[[M_C]]$ .

This ring is defined as  $H_*^{G_0}(\mathcal{R}_{G,N})$  equivariant homology

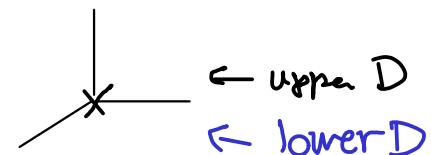
$\mathcal{R}_{G,N}$  = variety of triples

$= \{(P, \varphi, s) \mid \begin{array}{l} P: G\text{-bundles over } D \text{ (formal cpt 1-dim'l disk)} \\ \varphi: P|_{D^*} \xrightarrow{\sim} D^* \times G \text{ framing } D^* = D \setminus \{0\} \\ s: \text{section of } P \times_N N \text{ st. } \varphi(s) \text{ extends to } D \end{array}\}$

$G_0 = G(O) = \text{Map}(D, G)$  (change of  $\varphi$ )

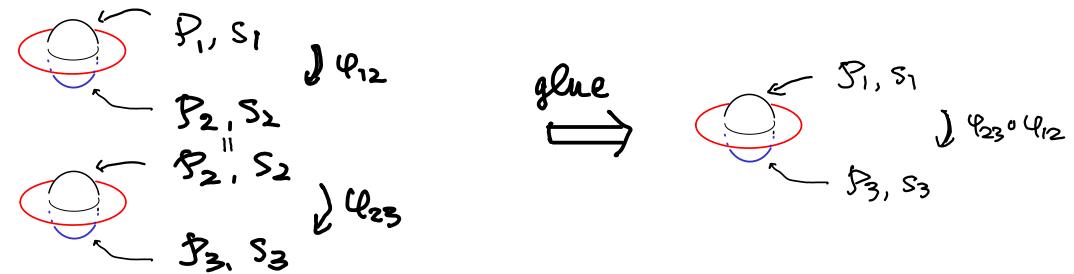


This is an algebraic-geometric caricature of a point singularity in  $\mathbb{R}^3$



multiplication  $H_*^{GO}(\mathcal{R}_{G,N}) \otimes H_*^{GO}(\mathcal{R}_{G,N}) \rightarrow H_*^{GO}(\mathcal{R}_{G,N})$

is given by **Convolution**.



Theorem  $H_*^{GO}(\mathcal{R}_{G,N})$  is a commutative ring.  $=: \mathbb{C}[M_C]$

Symplectic form (Poisson structure) is induced from the natural deformation quantization of  $H_*^{GO}(\mathcal{R}_{G,N})$  given by the extra  $\mathbb{C}^\times$ -action. (loop rotation)

Classical  $M_C$  is understood  $H_*^{GO}(\mathcal{R}_{G,N}^T)$  via Localization theorem.

Next : What should we do for Coulomb branches in mathematics ?

COULOMB BRANCHES OF 3D  $N=4$

QUIVER GAUGE THEORIES AND  
AFFINE GRASSMANNIAN SLICES

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### § 3. Geometric Satake for Kac-Moody Lie Algebras

$Q = (Q_0, Q_1)$  : quiver without edge loops

$\cdots \rightarrow V_2 \rightarrow V_1 \quad V_i, W_i \ (i \in Q_0)$  : finite dimensional cpx vector spaces

$$\begin{matrix} \cdots & \xrightarrow{\hspace{1cm}} & V_2 & \xrightarrow{\hspace{1cm}} & V_1 \\ & & \uparrow & & \uparrow \\ \cdots & & W_2 & & W_1 \end{matrix}$$

$$G = \prod GL(V_i)$$

$$N = \bigoplus_{\alpha \in Q_1} \text{Hom}(V_{0(\alpha)}, V_{i(\alpha)}) \oplus \bigoplus_i \text{Hom}(W_i, V_i)$$

$\mathfrak{g}$  = Kac-Moody Lie algebra with Cartan matrix  $a_{ij} = \begin{cases} 2 & (i=j) \\ -(\# \overset{\circ}{\rightarrow}_i^j + \# \overset{\circ}{\rightarrow}_j^i) & (i \neq j) \end{cases}$

$$\lambda = \sum \dim W_i \cdot \check{\alpha}_i, \quad \mu = \lambda - \sum \dim V_i \cdot \check{\alpha}_i$$

coweights of  $\mathfrak{g}$

$$M_C(G, N) \equiv M_C(\lambda, \mu)$$

Remark  $\equiv$  variant of construction for non-symmetric  
 (Nakajima-Weekes) Kac-Moody Lie algebras

Th [BFN : symmetric , NW : non-symmetric ]

Suppose  $\mathfrak{g}$  : finite dimensional complex simple Lie algebra

$\overline{\text{Gr}} \equiv \text{Gr}_G = \text{affine Grassmannian for } G$  s.t  $\text{Lie } G = \mathfrak{g}$  adjoint type

$$\overline{\text{Gr}^\lambda} = \overline{G_0 \cdot z^\lambda} \text{ (Schubert variety)} \supset \text{Gr}^\mu \quad \mu \leq \lambda$$

If  $\mu$ : dominant

$$\Rightarrow M_C(\lambda, \mu) \cong \text{slice to } \text{Gr}^\mu \text{ in } \overline{\text{Gr}^\lambda}$$

Remark

Even if  $\mu$ : not dominant,  $\cong$  similar description of  $M_C(\lambda, \mu)$   
(generalised slice)

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$T \subset G$  maximal torus is identified with  $\pi_1(G)^\wedge$  ← Pontryagin dual

$\pi_1(G)^\wedge \curvearrowright M_C(\lambda, \mu)$  is identified with one induced from  
 $T \curvearrowright \text{Gr}_G$

# CONJECTURAL GEOMETRIC SATAKE FOR KAC-MOODY

{ Proved for — finite dim,      usual geometric Satake + V. Krylov  
                  — affine type A      NAKAJIMA 2018

(0)  $M_C(\lambda, \mu)^T = \text{pt} \text{ or } \emptyset$

(1) ASSUME  $= \text{pt}$   
Choose  $x = \prod \det : G \rightarrow \mathbb{C}^\times$   
and consider  $\pi_1(x)^\wedge : \mathbb{C}^\times \rightarrow \pi_1(G)^\wedge = T$

DEFINE  $A_x(\lambda, \mu) := \{x \in M_C(\lambda, \mu) \mid \lim_{t \rightarrow 0} x(t)x \text{ exists}\}$   
(attracting set)

THEN  $A_x(\lambda, \mu) \cap M_C^{\text{reg}}(\lambda, \mu)$  is Lagrangian

(2)  $\bigoplus_{\mu} H_{\text{top}}(\mathcal{M}_c(\lambda, \mu))$  has a structure of  $T(\lambda)$   
integrable highest weight representation of  $\mathfrak{g}^{\vee}$

### REMARK

- The above formulation is an informal version.  
The correct version is in terms of intersection cohomology and  
hyperbolic localization as in Mirkovic-Vilonen's  
formulation of geometric Satake

$$\circ IH_{T,\text{cpt}}^*(\mathcal{M}_c(\lambda, \mu)) \longrightarrow H_{T_{\mathbb{Q}}}^*(\text{hyperbolic restriction of } IC(\mathcal{M}_c(\lambda, \mu)))$$

||  

$$(T(\lambda) \otimes \mathbb{C}[(\mathfrak{g}/\mathfrak{u})^*]^B \otimes \mathbb{C}_{-\mu})^B$$

$$G \supset B \supset U$$

$$\mathbb{V}_{\mu}(\lambda) \otimes_{\mathbb{C}} H_T^*(pt)$$

finite dim: Ginzburg - Riche  
affine type A : Muttiati - Nakajima (to appear)